Chair of Network Architectures and Services
School of Computation, Information and Technology
Technical University of Munich

Network Coding
CITHN2002, SoSe 2024

Tutorial 5

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Problem 1 Lossy wireless networks

We consider the three-node wireless relay network $G = (N, H)$ depicted in Figure 1 and the respective induced graph $G' = (N, A)$ in the lossy hypergraph model with orthogonal media access. For clarity, only the maximum hyperarcs are drawn in the figure.

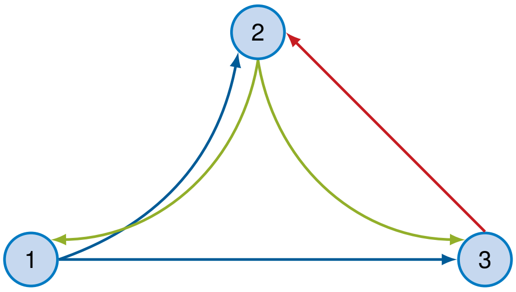


Figure 1: Three-node relay network

Hint: You may use the pre-printed Table 1 at the end of this problem sheet.

a)* Explicitly state the set of hyperarcs $\mathcal{H}$.

See Table 1 coumn 1.

b) State the set of hyperarc indices H by numbering the hyperarcs $(a, B) \in \mathcal{H}$ in lexicographic ascending order, i. e., $(a, B) < (a', B')$ if

1. $a < a'$ *or*
2. $a = a' \wedge |B| < |B'|$ *or*
3. $a = a' \wedge |B| = |B'| \wedge \min B \setminus B' < \min B' \setminus B$,

such that $j \equiv (a, B)$ with $j \in H = \{1, 2, ...\}$ for all $(a, B) \in \mathcal{H}$.

See Table 1 coumn 2.

c)*

For each $j \equiv (a, B) \in \mathcal{H}$, state the set $\mathcal{A}_j$ of arcs and corresponding indices A_j induced by j. Number the arcs $(a, b) \in \mathcal{A}$ in lexicographic ascending order, i. e., $(a, b) < (a', b')$ if

1. $a < a'$ *or*
2. $a = a' \wedge b < b'$,

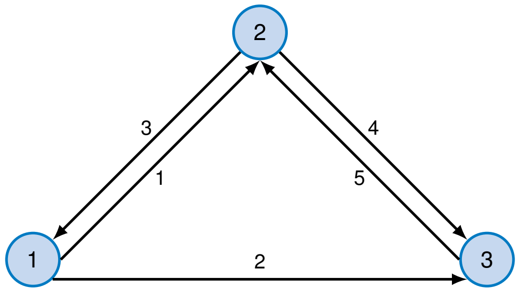
such that $k \equiv (a, b)$ with $k \in \{1, 2, ... \}$ for all $(a, b) \in \mathcal{A}$.

See Table 1 coulmn 3 and 4.

Prof. Dr.-Ing. Georg Carle
carle@tum.de

Stephan Günther
nc@net.in.tum.de

d) Draw the graph $G' = (N, A)$ that is induced by G.



(Numbers next to arcs denote the arc index $k \equiv (a, b) \in \mathcal{A}$.)

See solution of c), fourth column.

e) State the hyperarc-arc incidence matrix $\boldsymbol{N}$.

$$\boldsymbol{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

f) State the incidence matrix $\boldsymbol{M}$ for G'.

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

Assume that each arc $k \in A$ has unit capacity and a link error probability of $0 \leq \epsilon_k \leq 1$.

g) Determine the hyperarc capacity region

$$\mathcal{Z} = \bigcup_{\substack{\boldsymbol{\tau} \geq \boldsymbol{0} \\ \boldsymbol{1}^T \boldsymbol{\tau} \leq 1}} \left\{ \boldsymbol{z} : z_j = \tau_a \prod_{k \in A_j} (1 - \epsilon_k) \prod_{(a,b) \equiv k \notin A_j} \epsilon_k \quad \forall j \equiv (a, B) \in \mathcal{H} \right\}.$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_7 \end{bmatrix} = \begin{bmatrix} \tau_1(1-\epsilon_1)\epsilon_2 \\ \tau_1(1-\epsilon_2)\epsilon_1 \\ \tau_1(1-\epsilon_1)(1-\epsilon_2) \\ \tau_2(1-\epsilon_3)\epsilon_4 \\ \tau_2(1-\epsilon_4)\epsilon_3 \\ \tau_2(1-\epsilon_3)(1-\epsilon_4) \\ \tau_3(1-\epsilon_5) \end{bmatrix}$$

h) State the hyperarc-hyperarc incidence matrix $\boldsymbol{Q}$.

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

i) Determine the broadcast capacity vector $\boldsymbol{y}$.

$$\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{z} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} z_1 + z_3 \\ z_2 + z_3 \\ z_1 + z_2 + z_3 \\ z_4 + z_6 \\ z_5 + z_6 \\ z_4 + z_5 + z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} \tau_1(1-\epsilon_1) \\ \tau_1(1-\epsilon_2) \\ \tau_1(1-\epsilon_1\epsilon_2) \\ \tau_2(1-\epsilon_3) \\ \tau_2(1-\epsilon_4) \\ \tau_2(1-\epsilon_3\epsilon_4) \\ \tau_3(1-\epsilon_5) \end{bmatrix}$$

j) Explicitly state the lossy hyperarc flow bound $\boldsymbol{N}\boldsymbol{x} \leq \boldsymbol{y}$.

$$\boldsymbol{N}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \\ x_3 \\ x_4 \\ x_3 + x_4 \\ x_5 \end{bmatrix} \leq \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} z_1 + z_3 \\ z_2 + z_3 \\ z_1 + z_2 + z_3 \\ z_4 + z_5 \\ z_5 + z_6 \\ z_4 + z_5 + z_6 \\ z_7 \end{bmatrix}$$

k) Enumerate all $s - t$ cuts S and their respective capacities $v(S_a)$ for $s = 1$ and $t = 3$.

$$\begin{aligned} S_1 &= \{1\} \\ S_2 &= \{1, 2\} \\ v(S_1) &= y_3 = z_1 + z_2 + z_3 \\ &= \tau_1\left((1-\epsilon_1)\epsilon_2 + (1-\epsilon_2)\epsilon_1 + (1-\epsilon_1)(1-\epsilon_2)\right) \\ &= \tau_1(1-\epsilon_1\epsilon_2) \\ v(S_2) &= y_2 + y_5 = z_2 + z_3 + z_5 + z_6 \\ &= \tau_1\left((1-\epsilon_2)\epsilon_1 + (1-\epsilon_1)(1-\epsilon_2)\right) + \tau_2\left((1-\epsilon_4)\epsilon_3 + (1-\epsilon_3)(1-\epsilon_4)\right) \\ &= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4) \end{aligned}$$

l) State the min-cut capacity r for a flow from s to t in dependency of τ_1 and τ_2.

$$r = \min\{v(S_1), v_(S_2)\} = \min\{\tau_1(1-\epsilon_1\epsilon_2), \tau_1(1-\epsilon_2)+\tau_2(1-\epsilon_4)\}$$

m) Determine τ_1 and τ_2 such that r is maximized.

We need to solve the optimization problem

$$r^* = \max_{\substack{\tau_1,\tau_2\geq 0\\ \tau_1+\tau_2=1}} \min\{\tau_1(1-\epsilon_1\epsilon_2), \tau_1(1-\epsilon_2)+\tau_2(1-\epsilon_4)\}.$$

In case that $v(S_1) \neq v(S_2)$ we will increase the smaller one, which might decrease the larger one. The optimal solution is found when we either cannot further increase the value of the smaller cut or when $v(S_1) = v(S_2)$.

From the induced graph (see solution of (d)) we see that node 2 cannot contribute if $\epsilon_4 > \epsilon_2$. In this case only node 1 will transmit and thus $\tau_1 = 1$ and $\tau_2 = 0$. The same is obviously true when $\epsilon_1 = 1$ since node 2 cannot receive anything from node 1 in this case.
For $\epsilon_4 \leq \epsilon_2$, $\epsilon_1 < 1$, and $\tau_1 = 1$ we find that $v(S_1) > v(S_2)$. We therefore increase τ_2 at the cost of τ_1 until $v(S_1) = v(S_2)$, which is the optimal solution:

$$\begin{aligned}
\tau_1 + \tau_2 = 1 \quad &\Rightarrow \quad \tau_2 = 1 - \tau_1 \\
v(S_1) &= \tau_1(1-\epsilon_1\epsilon_2) \\
v(S_2) &= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4) \\
&= \tau_1(\epsilon_4-\epsilon_2) + 1 - \epsilon_4 \\
v(S_1) &\stackrel{!}{=} v(S_2) \\
\tau_1(1-\epsilon_1\epsilon_2) &= \tau_1(\epsilon_4-\epsilon_2) + 1 - \epsilon_4 \\
\tau_1(1-\epsilon_4-\epsilon_1\epsilon_2+\epsilon_2) &= 1-\epsilon_4 \\
\tau_1 &= \frac{1-\epsilon_4}{1-\epsilon_4-\epsilon_1\epsilon_2+\epsilon_2}
\end{aligned}$$

We therefore get the following solution:

$$\tau_1 = \begin{cases} 1 & \epsilon_1 = 1 \vee \epsilon_2 \leq \epsilon_4, \\ \dfrac{1-\epsilon_4}{1-\epsilon_4-\epsilon_1\epsilon_2+\epsilon_2} & \epsilon_2 > \epsilon_4. \end{cases}$$

Note that we could modify the cases such that $\epsilon_2 < \epsilon_4$ and $\epsilon_2 \geq \epsilon_4$ without affecting the capacity.

We now consider the multicast $s = 1$ and $T = \{2, 3\}$.

n) Determine the missing $s - T$ cut and its capacity.

$S_3 = \{1, 3\}$ with

$$v(S_3) = y_1 + y_7 = z_1 + z_3 + z_7 = \tau_1(1-\epsilon_1) + \tau_3(1-\epsilon_5)$$

o) State the optimization problem to maximize the multicast capacity r'.

$$\max_{\substack{\boldsymbol{\tau} \geq \mathbf{0} \\ \mathbf{1}^T \boldsymbol{\tau} = 1}} \min \{v(S_1), v(S_2), v(S_3)\}$$

p) Determine the maximum multicast rate r'^* by solving the problem. Assume $\epsilon_4 = \epsilon_5$, otherwise the various cases are more complex.

Hint: It is sufficient to differentiate between cases and to express τ_2, τ_3 by means of τ_1. Except for the trivial case, the expression for τ_1 is not nice.

$$\begin{aligned} \tau_1 + \tau_2 + \tau_3 &= 1 \\ v(S_1) &= \tau_1(1 - \epsilon_1\epsilon_2) \\ v(S_2) &= \tau_1(1 - \epsilon_2) + \tau_2(1 - \epsilon_4) \\ v(S_3) &= \tau_1(1 - \epsilon_1) + \tau_3(1 - \epsilon_5) \end{aligned}$$

There is no valid solution, yet . . . – if somebody finds one, please notify us.

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carle@tum.de

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$(a, B) \in \mathcal{H}$	$j \equiv (a, B)$	$\mathcal{A}_j$	A_j	z_j	y_j
(1,{2})	1	$\{(1,2)\}$	$\{1\}$	$\tau_1(1-\epsilon_1)\epsilon_2$	$\tau_1(1-\epsilon_1)$
(1,{3})	2	$\{(1,3)\}$	$\{2\}$	$\tau_1(1-\epsilon_3)\epsilon_1$	$\tau_1(1-\epsilon_2)$
(1,{2,3})	3	$\{(1,2),(1,3)\}$	$\{1,2\}$	$\tau_1(1-\epsilon_1)(1-\epsilon_2)$	$\tau_1(1-\epsilon_1\epsilon_2)$
(2,{1})	4	$\{(2,1)\}$	$\{3\}$	$\tau_2(1-\epsilon_3)\epsilon_4$	$\tau_2(1-\epsilon_3)$
(2,{3})	5	$\{(2,3)\}$	$\{4\}$	$\tau_2(1-\epsilon_4)\epsilon_3$	$\tau_2(1-\epsilon_4)$
(2,{1,3})	6	$\{(2,1),(2,3)\}$	$\{3,4\}$	$\tau_2(1-\epsilon_3)(1-\epsilon_4)$	$\tau_2(1-\epsilon_3\epsilon_4)$
(3,{2})	7	$\{(3,2)\}$	$\{5\}$	$\tau_3(1-\epsilon_5)$	$\tau_3(1-\epsilon_5)$

Table 1: Fill in values from different subproblems.