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Network Coding
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Tutorial 4

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Problem 1 Maximum flow problem

We consider the wired network with $n = 6$ nodes and $m = 7$ arcs that is described by the incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

Arcs are enumerated in lexicographic order as known from the lecture, e. g. $(1, 2) \prec (2, 1)$.

a)* Draw the network described by $\boldsymbol{M}$ and label both nodes and arcs by their indices.

b) What is the rank $\boldsymbol{M}$?

c)* Determine a basis B of null $\boldsymbol{M}$.

The arc capacities are given by $\boldsymbol{z} = [2\ 2\ 2\ 2\ 1\ 2\ 2]^T$. We consider a single unicast between nodes 1 and 6 described by $\boldsymbol{d} = [1\ 0\ 0\ 0\ 0\ -1]^T$.

d)* Determine the capacity between nodes 1 and 6 by using the min-cut / max-flow theorem. (A bit tedious to enumerate all the cuts . . .)

The maximum flow problem is formally expressed as linear program

$$\max_{r,\boldsymbol{x}} \ r \quad \text{s.t.} \quad \boldsymbol{Mx} = r\boldsymbol{d}, \tag{1}$$
$$\boldsymbol{x} \geq \boldsymbol{0}, \tag{2}$$
$$\boldsymbol{x} \leq \boldsymbol{z}. \tag{3}$$

In order to solve this problem using Matlab we have to rewrite it as

$$\min_{\boldsymbol{x}} \ \boldsymbol{f}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{Ax} \leq \boldsymbol{a}, \tag{4}$$
$$\boldsymbol{Bx} = \boldsymbol{b}, \tag{5}$$
$$\boldsymbol{x} \geq 0, \tag{6}$$
$$\boldsymbol{x} \leq \boldsymbol{z}. \tag{7}$$

e)* Express the scalar rate r by means of $\boldsymbol{M}$, $\boldsymbol{x}$, and $\boldsymbol{d}$.

f) Determine $\boldsymbol{B}$ such that $\boldsymbol{Bx} = \boldsymbol{0}$ is equivalent to $\boldsymbol{Mx} = r\boldsymbol{d}$.

g) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r$.

h) State the revised optimization problem and solve it using Matlab.

Now we consider the same problem but with two competing flows, i. e., we have two demand vectors

$$\boldsymbol{d}_1 = [1\ 0\ 0\ 0\ \ -1\ 0]^T \text{ and}$$
$$\boldsymbol{d}_2 = [0\ 1\ 0\ 0\ 0\ \ -1]^T.$$

If we want to maximize the joint rate $r = r_1 + r_2$, the optimization problem becomes

$$\begin{aligned}\max_{r_1,r_2}\ r_1 + r_2 \quad \text{s. t.} \quad \boldsymbol{Mx}_1 &= r_1\boldsymbol{d}_1,\\ \boldsymbol{Mx}_2 &= r_2\boldsymbol{d}_2,\\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq \boldsymbol{0},\\ \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq \boldsymbol{z}.\end{aligned}$$

We now restate this problem to solve it in Matlab. To this end, we define

$$\boldsymbol{N} = \begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{M}\end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix},\ \boldsymbol{r} = \begin{bmatrix}r_1\\ r_2\end{bmatrix},\ \text{and}\ \ \boldsymbol{D} = \begin{bmatrix}\boldsymbol{d}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{d}_2\end{bmatrix}.$$

i) Express $\boldsymbol{r}$ by $\boldsymbol{N}$, $\boldsymbol{D}$, and $\boldsymbol{x}$.
Hint: For any matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ with linear independent columns holds that $\boldsymbol{AA}^T \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A}^T\boldsymbol{A} \in \mathbb{R}^{m\times m}$ are of rank n and m, respectively, i. e., $\left(\boldsymbol{AA}^T\right)^{-1}$ and $\left(\boldsymbol{A}^T\boldsymbol{A}\right)^{-1}$ exist.

j) Determine $\boldsymbol{B}$ such that $\boldsymbol{Bx} = \boldsymbol{0}$ is equivalent to $\boldsymbol{Nx} = \boldsymbol{Dr}$.

k) Determine $\boldsymbol{A}$ such that $\boldsymbol{Ax} \leq \boldsymbol{z}$ describes the joint capacity constraint.

l) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r_1 + r_2$.

m) State the final problem and solve it in Matlab.

n) Sketch the achievable rate region.