Chair of Distributed Systems and Security
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# Network Coding
**IN2315, SoSe 2024**

**Tutorial 1**

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## Problem 1 DCF contention window, collision probabilities, and MAC fairness

The *Distributed Coordination Function (DCF)* essentially represents the most basic form of CSMA/CA:

1. Listen before talk, i. e., sense the medium and wait until the medium becomes idle.
2. In contrast to CD, do not start transmitting at the beginning of the next time slot but do *collision avoidance*:
   (a) From the set $\mathcal{C} = \{0, 1, \ldots, C\}$, draw independentely and uniformely distributed a random number $c \in \mathcal{C}$ of contention slots.
   (b) Wait for $c$ slot times and sense the medium again.
   (c) If the medium is still free, start transmitting at the beginning of the next slot. Start allover again otherwise.

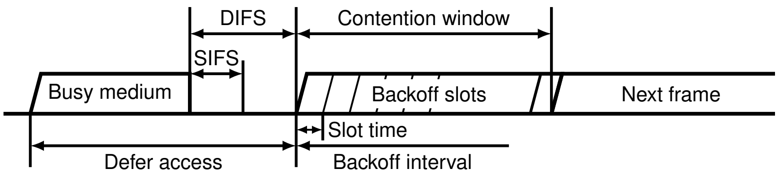


**Figure 1:** DCF [1, Section 9.3.2.3.1]

The above description is a bit simplified, i. e., in reality nodes do not simply wait for $c$ slot times but keep sensing the medium and suspend counting down while the medium becomes idle. For simplicity, we ignore that and assume the DCF works as described above. Furthermore, we assume that nodes are synchronized.

In the following we assume a two-node network. Both nodes are backlogged, i. e., have data to transmit. Each node maintains a contention window $\mathcal{C}_i$, $i \in \{1, 2\}$ determined by its maximum value $C_i$. The medium is assumed to be initially idle.

**a)*** Determine the probability $\Pr[X = n]$ that a node draws a backoff period of exactly $n$ slot times.

$$\Pr[X = n] = \frac{1}{C + 1}$$

**b)*** Determine the probability $\Pr[X \leq n]$ that a node draws a backoff period of at most $n$ slot times.

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$$
\begin{aligned}
\Pr[X \leq n] &= \sum_{k=0}^{n} \Pr[X = k] \\
&= \sum_{k=0}^{n} \frac{1}{C+1} \\
&= \frac{n+1}{C+1}
\end{aligned}
$$

**c)*** Determine the probability of a collision if $C_1 = C_2$.

$$
\begin{aligned}
\Pr[X_1 = X_2] &= \sum_{k=0}^{C} \Pr[X_1 = k \wedge X_2 = k] \\
&= \sum_{k=0}^{C} \Pr[X_1 = k] \Pr[X_2 = k] \\
&= \sum_{k=0}^{C} \left(\frac{1}{C+1}\right)^2 \\
&= \frac{1}{C+1}
\end{aligned}
$$

**d)** Determine the probability of a collision if $C_1 \neq C_2$.

$$
\begin{aligned}
\Pr[X_1 = X_2] &= \sum_{k=0}^{\min\{C_1,C_2\}} \Pr[X_1 = k \wedge X_2 = k] \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \Pr[X_1 = k] \Pr[X_2 = k] \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \frac{1}{C_1+1} \frac{1}{C_2+1} \\
&= \frac{\min\{C_1, C_2\} + 1}{(C_1+1)(C_2+1)} \\
&= \frac{1}{\max\{C_1, C_2\} + 1}
\end{aligned}
$$

**e)** Determine the probability that node 1 wins the contention phase assuming that $C_1 \leq C_2$.

$$
\begin{aligned}
\Pr[X_1 < X_2] &= \sum_{k=0}^{C_1} \Pr[X_1 = k \wedge X_2 > k] \\
&= \sum_{k=0}^{C_1} \Pr[X_1 = k] \Pr[X_2 > k] \\
&= \sum_{k=0}^{C_1} \frac{1}{C_1 + 1} \left(1 - \frac{k+1}{C_2 + 1}\right) \\
&= \sum_{k=0}^{C_1} \frac{C_2 - k}{(C_1 + 1)(C_2 + 1)} \\
&= \sum_{k=0}^{C_1} \frac{C_2}{(C_1 + 1)(C_2 + 1)} - \sum_{k=0}^{C_1} \frac{k}{(C_1 + 1)(C_2 + 1)} \\
&= \frac{C_2(C_1 + 1)}{(C_1 + 1)(C_2 + 1)} - \frac{C_1(C_1 + 1)}{2(C_1 + 1)(C_2 + 1)} \\
&= \frac{2C_2 - C_1}{2(C_2 + 1)}
\end{aligned}
$$

**f)*** Determine the expected duration of the contention phase for an $n$-node network assuming that $C_i = C$ for $i = 1, \dots, n$.

$$
\begin{aligned}
\mathrm{E}[T_n] &= \mathrm{E}[\min\{X_i | i \in \{1, \dots, n\}\}] \\
&=^1 \sum_{k=0}^{C} \Pr[X_i > k | i \in \{1, \dots, n\}] \\
&= \sum_{k=0}^{C} \prod_{i=1}^{n} \Pr[X_i > k] \\
&= \sum_{k=0}^{C} \prod_{i=1}^{n} (1 - \Pr[X_i \leq k]) \\
&= \sum_{k=0}^{C} \prod_{i=1}^{n} \left(1 - \frac{k+1}{C+1}\right) \\
&= \sum_{k=0}^{C} \left(1 - \frac{k+1}{C+1}\right)^n \\
&= \sum_{k=0}^{C} \left(\frac{C-k}{C+1}\right)^n \\
&= \sum_{k=0}^{C} \left(\frac{k}{C+1}\right)^n \\
&= \frac{1}{(C+1)^n} \sum_{k=0}^{C} k^n
\end{aligned}
$$

[1]Remember that $\mathrm{E}[X] = \sum_{k=0}^{\infty} k \Pr[X = k] = \sum_{k=0}^{\infty} (1 - \Pr[X \leq k]) = \sum_{k=0}^{\infty} \Pr[X > k]$.

**g)** Determine the expected duration of the contention phase for a 2-node network assuming that $C_1 \neq C_2$.

$$
\begin{aligned}
\mathrm{E}[T] &= \mathrm{E}[min\{X_1, X_2\}] \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \Pr[X_1 > k \wedge X_2 > k] \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \Pr[X_1 > k] \Pr[X_2 > k] \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} (1 - \Pr[X_1 \leq k])(1 - \Pr[X_2 \leq k]) \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \left(1 - \frac{k+1}{C_1+1}\right)\left(1 - \frac{k+1}{C_2+1}\right) \\
&= \sum_{k=0}^{\min\{C_1,C_2\}} \frac{C_1 - k}{C_1 + 1} \frac{C_2 - k}{C_2 + 1} \\
&= \frac{1}{(C_1+1)(C_2+1)} \sum_{k=0}^{\min\{C_1,C_2\}} (C_1 - k)(C_2 - k) \\
&=^1 \frac{1}{(C_1+1)(C_2+1)} \sum_{k=0}^{\min\{C_1,C_2\}} k(k + |C_1 - C_2|) \\
&= \frac{1}{(C_1+1)(C_2+1)} \left( \sum_{k=0}^{\min\{C_1,C_2\}} k^2 + \sum_{k=0}^{\min\{C_1,C_2\}} k\,|C_1 - C_2| \right) \\
&=^2 \frac{\min\{C_1,C_2\}\,(\min\{C_1,C_2\}+1)}{2(C_1+1)(C_2+1)} \left( \frac{2\min\{C_1,C_2\}+1}{3} + |C_1 - C_2| \right)
\end{aligned}
$$

[1] Note that the sum goes from 0 to $\min\{C_1, C_2\}$. Just assume that $C_1 < C_2$. Then, $C_1 - k$ ranges from 0 to $C_1$, while $C_2 - k$ ranges from $C_2 - C_1$ to $C_2$. Generalization should now be easy.

[2] Simply set $\sum_{k=1}^{n} k = n(n+1)/2$ and $\sum_{k=1}^{n} k^2 = n(n+1)(n+2)/6$ and factor out.

## Problem 2 Matlab

Download Matlab from `https://matlab.rbg.tum.de` It is free for TUM students. Using Matlab, plot the probabilities of Problem 1 for $0 < C < 16$.

## References

[1] "IEEE Standard for Information Technology, telecommunications and Information Exchange between Systems, local and Metropolitan Area Networks, Specific Requirements Part 11: Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications," 2012.