Network Coding (NC)

CITHN2002 – Summer 2024

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Technical University of Munich

Chapter 5: Models

Networks as graphs

Flow problems

Minimum cost flow problem

Maximum s-t flow problem

Min-cut and its capacity

Minimum cost maximum s–t flow problem

Multicommodity flow problems

Multicast in networks

Store-forward multicast

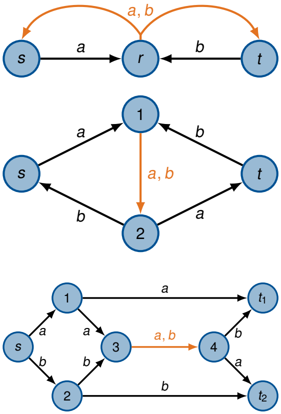
Multicast tree-based forwarding

Multicast with network coding

Wireless Packet Networks

Networks as graphs

Networks as graphs



- (Wired) Networks can be modeled as abstract graphs.
- Information flow in networks with routing and forwarding can be modeled as (multi-)commodity flow problem.
- Gives nice problems (flow optimization problems) and algorithms (Dijkstra, Bellman-Ford, etc.).
- Special properties of “Information” (arbitrarily reproducible, coded representation, etc.) are not taken into account in the standard commodity model.

Networks as graphs

- The set of nodes is given by $N = \{1, \dots, n\}$
- The set of arc indices is given by $A = \{1, 2, \dots, m\}$
  - Each arc index $j \in A$ represents an ordered pair of nodes
  - We therefore write $j \equiv (a, b)$
- The set of arcs is given by $\mathcal{A} = \{(a, b) \mid \exists$ link from $a \in N$ to $b \in N\}$

Important structures

- path (directed, undirected)
- tree (directed, undirected)
- cycle (directed, undirected)

Note: We assume $\mathcal{G}$ is connected, i. e., there exists an undirected path between any pair of nodes.

Networks as graphs

Examples

- Enumeration of arcs is arbitrary but must be fixed for a given network.
- Convention: use lexicographic order, i. e., $(2, 1) \prec (2, 3) \prec (3, 2)$.

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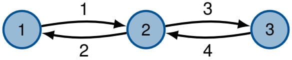

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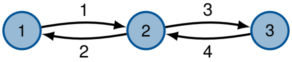

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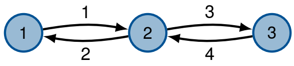

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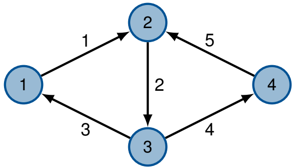

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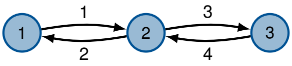

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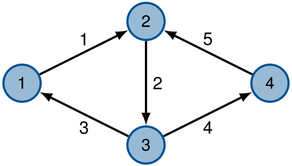

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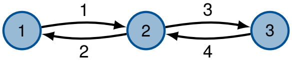

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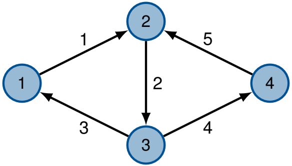

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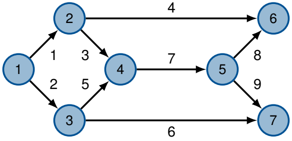

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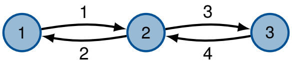

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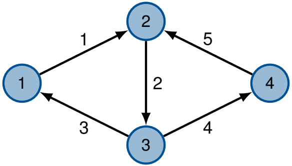

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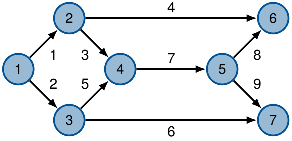

$$N = \{1, 2, 3, 4, 5, 6, 7\}$$
$$\mathcal{A} = \{(1,2), (1,3), (2,4), (2,6), (3,4), (3,7), (4,5), (5,6), (5,7)\}$$

Networks as graphs

Definition: incidence matrix $\boldsymbol{M}$

Given $\mathcal{G} = (N, A)$, we define the incidence matrix $\boldsymbol{M} = (m_{ij}) \in \{-1, 0, 1\}^{|N| \times |A|}$ where $\forall i \in N$ and $j \in A$

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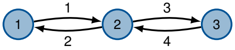

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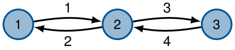

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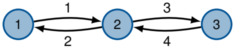

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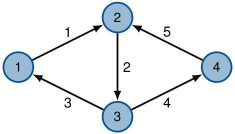

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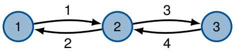

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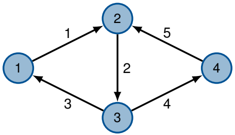

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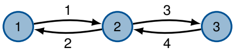

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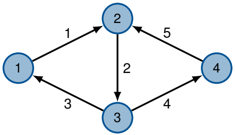

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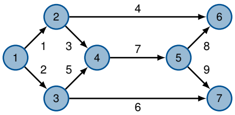

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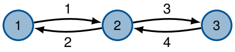

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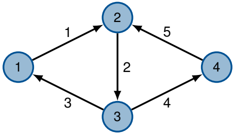

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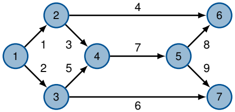

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Networks as graphs

Definition: undirected cycles in connected graphs

An undirected cycle $C \subset A$ is defined as vector $\boldsymbol{c} \in \{-1, 0, 1\}^{|A|}$ where

$$c_j = \begin{cases} 1 & \text{if } j \text{ is traversed in forward direction,} \\ -1 & \text{if } j \text{ is traversed in backward direction,} \\ 0 & \text{otherwise.} \end{cases}$$

The set of all cycles is denoted by $\mathcal{C}$.

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2 Proof via undirected tree in G, adding any further arc creates a cycle

3 Number of linearly independent undirected cycles

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$$\text{null}\, \boldsymbol{M}^\mathsf{T} = \text{span}\{\mathbf{1}\}^1$$

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Dimensions:

- rank $\boldsymbol{M} = n - 1$ [2]
- dim null $\boldsymbol{M}^\mathsf{T} = 1$
- dim null $\boldsymbol{M} = m - n + 1$ [3]

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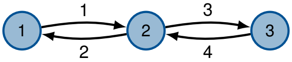
Examples: nullspace and cycles



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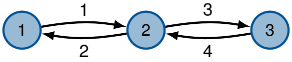

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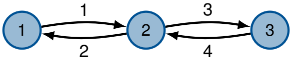

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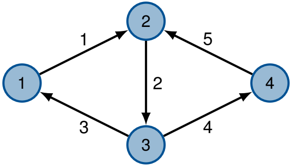

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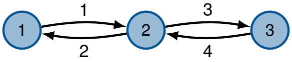

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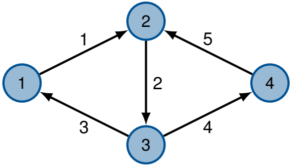

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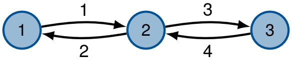

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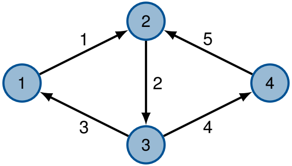

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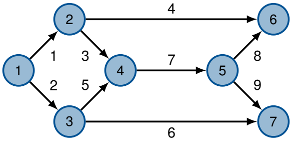

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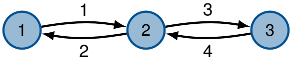

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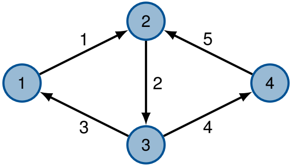

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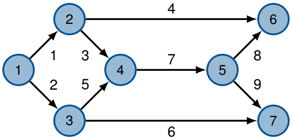

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Wireless Packet Networks

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- The flow vector $\boldsymbol{x} = [x_1, \dots, x_m]^\mathsf{T}$ represents the amount of commodity (information) on each arc.
- The source vector $\boldsymbol{d} = [d_1, \dots, d_n]^\mathsf{T}$ represents the amount of commodity (information) that any node injects or consumes.
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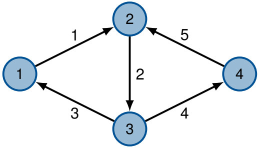
- FCL contains exactly one redundant constraint since rank $\boldsymbol{M} = n - 1$ (if graph is connected).
- Flows along directed cycles are independent of $\boldsymbol{d}$, i. e., flows that satisfy $\boldsymbol{Mx} = \boldsymbol{0}$, $\boldsymbol{x} \geq \boldsymbol{0}$.

Example 1: Diamond network from $s = 1$ to $t = 4$



- Incidence matrix and source vector:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

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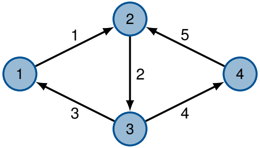

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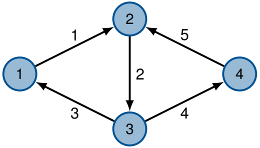

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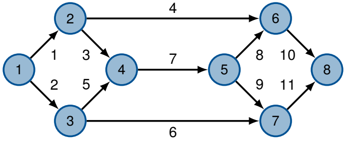
Example 2: Extended butterfly from $s = 1$ to $t = 8$



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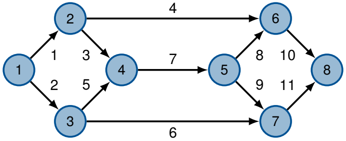

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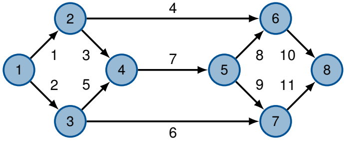

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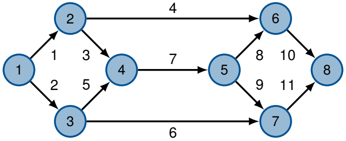

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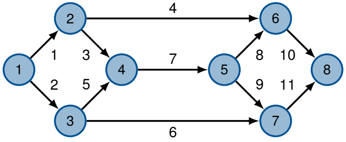

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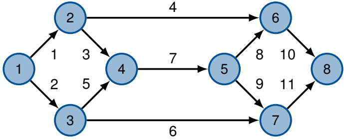

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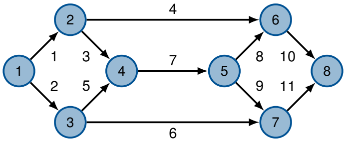

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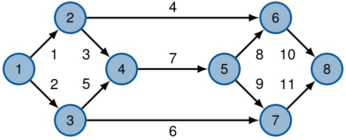

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Definition: feasible flow region

Given the incidence matrix $\boldsymbol{M}$ of a connected graph $\mathcal{G} = (N, A)$ and a source vector $\boldsymbol{d} \geq \mathbf{0}$, the feasible flow region is given by

$$\mathcal{F}(\boldsymbol{M}, \boldsymbol{d}) = \{\boldsymbol{x} : \boldsymbol{M}\boldsymbol{x} = \boldsymbol{d}, \boldsymbol{x} \geq 0\},$$

which is

- a closed[1] polyhedral[2] convex[3] set,
- nonempty if $\mathbf{1}^\top \boldsymbol{d} = 0$ (and G is connected),
- bounded[4] if G is acyclic (contains no directed cycles), i. e., $\mathcal{F}(\boldsymbol{M}, \mathbf{0}) = \{\mathbf{0}\}$,
- and, in general, contains infinitely many solutions.

1 A set $\mathcal{X}$ is closed if it contains all its limit points.
2 A set $\mathcal{X}$ is a polyhedron if it is defined by a finite number of affine (in)equalities, i.e., $\mathcal{X} = \{\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b}\}$.
3 A set $\mathcal{X}$ is convex if for any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$ and any real scalar $\lambda \in [0, 1]$, $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in \mathcal{X}$.
4 A set $\mathcal{X}$ is bounded if it is contained in some ball around the origin, i.e., $\mathcal{X} \subset B_r(\mathbf{0})$ for some $r > 0$.

Minimum cost flow problem

Uncapacitated minimum cost flow problem

Cost per unit flow on arcs: $\boldsymbol{c} = [c_1, \dots, c_m]^\mathsf{T}$

$$\begin{aligned} \min \quad & \boldsymbol{c}^\mathsf{T} \boldsymbol{x} \\ \text{s. t.} \quad & \boldsymbol{M}\boldsymbol{x} = \boldsymbol{d} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Capacitated minimum cost flow problem

$\boldsymbol{z} = [z_1, \dots, z_m]^\mathsf{T}$ maximum flow on each arc

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[5] Not all flow solutions to these two problems describe shortest paths, but at least one does.

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Example: Shortest path[5]

- $\boldsymbol{c}$ "length" of each arc, e.g., $\boldsymbol{c} = \boldsymbol{1}$ (number of hops metric)
- Shortest path from s to t: $d_s = 1$, $d_t = -1$, $d_i = 0 \ \forall\ i \neq s, t$
- Simultaneous shortest paths to t: $d_t = -n + 1$, $d_i = 1 \ \forall\ i \neq t$

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Solution approaches

- General purpose linear programming solver (Simplex, Interior point, etc.)
- Specialized algorithms (Dijkstra, Bellman-Ford, network simplex, etc.) exploiting graph structure and recursive structure of the optimal solution (if available)

Maximum s-t flow problem

Capacitated minimum cost flow problem

$\boldsymbol{z} = [z_1, \dots, z_m]^\mathsf{T}$ maximum flow on each arc with source vector $d_s = 1$, $d_t = -1$, $d_i = 0 \; \forall \; i \neq s, t$

$$\begin{aligned} \max \quad & r \\ \text{s. t.} \quad & \boldsymbol{Mx} = r\boldsymbol{d} \\ & \boldsymbol{x} \leq \boldsymbol{z} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Solution approaches

- General purpose linear programming solver (Simplex, Interior point, etc.)
- Lagrangian duality approaches (selectively relax one constraint)
- Specialized algorithms (Ford-Fulkerson) exploiting graph structure and relation to min-cut

Min-cut and its capacity

- An s–t cut is a subset of nodes $S \subset N$ such that $s \in S$ and $t \notin S$.
- An arc $(i, j) \in \mathcal{A}$ crosses S if $i \in S$ and $j \notin S$.
- $\mathcal{A}(S)$ denotes all crossing arcs.
- The value of an s–t cut given the capacity vector $\boldsymbol{z}$ is defined as

$$v(S) = \sum_{(i,j) \in \mathcal{A}(S)} z_{ij}.$$

- The value of any s–t cut upper bounds the maximum s–t flow.

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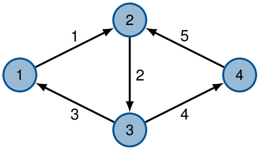
Max-flow min-cut theorem

The value of the minimum s–t cut equals the value of the maximum s–t flow, i.e.,

$$\max\{r : \boldsymbol{Mx} = r\boldsymbol{d},\ \boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{z}\} = \min\{v(S) : S \text{ is } s\text{–}t \text{ cut}\}.$$

Min-cut and its capacity

Example 1: Diamond network from $s = 1$ to $t = 4$



- Incidence matrix, source vector, capacity vector:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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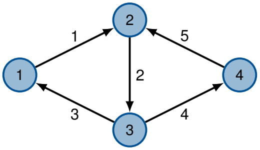

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- Max-flow:

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Min-cut and its capacity

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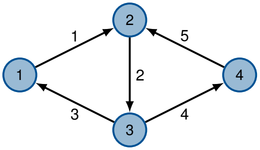

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$$\min\{v(S) : S \text{ is } s\text{-}t \text{ cut}\} = 1$$

Minimum cost maximum s–t flow problem

Generalizes (uncapacitated) minimum cost and (capacitated) maximum flow s–t problem:

- Source and flow vector: $\boldsymbol{d}$, $\boldsymbol{x}$
- Capacity and cost vector: $\boldsymbol{z}$, $\boldsymbol{c}$

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s. t.} \quad & \boldsymbol{M}\boldsymbol{x} = \boldsymbol{d} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \leq \boldsymbol{z} \end{aligned}$$

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Special cases

- Maximum s–t flow (see tutorial)
- Minimum cost flow (capacitated $\boldsymbol{z} < \infty$, uncapacitated $\boldsymbol{z} = \infty$)

Multicommodity flow problems

In contrast to single-commodity flow problems we now have multiple commodities, e. g. flows, that compete with each other:

- Commodities $C = \{1, \dots, c\}$,
- Source, flow, and cost vector of commodity k: $\boldsymbol{d}_k$, $\boldsymbol{x}_k$, $\boldsymbol{c}_k$
- Capacity shared across all commodities: $\boldsymbol{z}$

The min-cost max-flow problem then reads as:

$$\begin{aligned}
\min \quad & \sum_{k \in C} \boldsymbol{c}_k^T \boldsymbol{x}_k \\
\text{s. t.} \quad & \boldsymbol{M}\boldsymbol{x}_k = \boldsymbol{d}_k \quad \forall k \in C \\
& \boldsymbol{x}_k \geq \boldsymbol{0} \quad \forall k \in C \\
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\end{aligned}
$$

Properties

- Flow conservation applies to all commodities individually
- Capacity is shared among all commodities

Optimality of a solution now even more depends on what is considered “optimal”:

- The previous definition is a joint optimization of the weighted sum rate $\sum_k \boldsymbol{c}_k^\mathsf{T} \boldsymbol{x}_k$.
- This allows that commodities (flows) are assigned few or no resources at all.
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Solution approaches

- General purpose linear programming solver
- Lagrangian duality approaches (selectively relax one constraint, mostly the capacity constraint which couples all flows)

Networks as graphs

Flow problems

Multicast in networks

- Store-forward multicast
- Multicast tree-based forwarding
- Multicast with network coding

Wireless Packet Networks

Multicast in networks

Multicast in networks as flow problems

- Multicast communication is identified by its terminal set $T \subset N$.
- We can consider one or multiple sources (there is no big difference from a theoretical perspective).
- Special cases:

Multicast in networks

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  - unicast (one source, one terminal)
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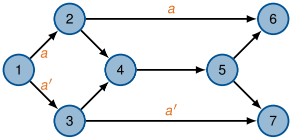
How is multicast treated in networks?

- Convert to unicasts
  $\rightarrow$ replicate packets at source and store-forward at all other nodes
- Allow replication at all nodes
  $\rightarrow$ multicast tree / Steiner tree based forwarding
- Allow coding at all nodes
  $\rightarrow$ network coding

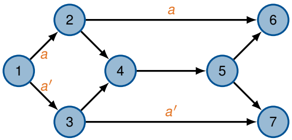
Store-forward multicast



- The flows to the terminals are independent of each other.
- Capacity needs to be split among all flows.

max s–T flow problem:

Store-forward multicast



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max s–T flow problem:

- One commodity for each terminal $t \in T$
- Source vector $\boldsymbol{d}_{st}$ such that $d_{st,s} = 1$, $d_{st,t} = -1$, and $d_{st,i} = 0$ otherwise
- Capacity vector $\boldsymbol{z}$ split among commodities

$$\max\ r \quad \text{s.t.} \quad \begin{aligned} \boldsymbol{M}\boldsymbol{x}_t &= r\boldsymbol{d}_{st} && \forall t \in T \\ \boldsymbol{x}_t &\geq \boldsymbol{0} && \forall t \in T \\ \sum_{t \in T} \boldsymbol{x}_t &\leq \boldsymbol{z} \end{aligned}$$

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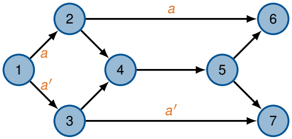

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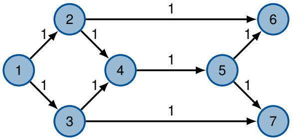
$\Rightarrow$ That is a multicommodity flow problem!

Store-forward multicast

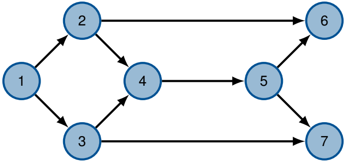


- Optimal flow solutions
  - $\boldsymbol{x}_6 = [1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0]^T$
  - $\boldsymbol{x}_7 = [0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0]^T$
- Total flow which is capacity relevant
  - $\boldsymbol{x}_6 + \boldsymbol{x}_7 = [1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0]^T$

maximum multicast s–T flow = 1

Multicast tree-based forwarding

- s–T multicast tree: a tree rooted at s such that there exists a directed path to each $t \in T$ (arcs belong to at least one path).
- Unit flow on multicast tree delivers one unit (the same unit) of information to each terminal.



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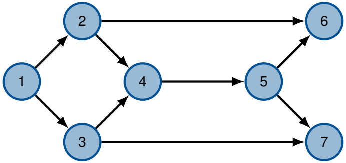

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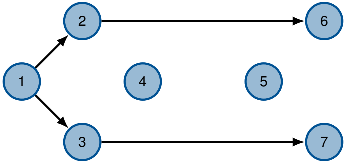

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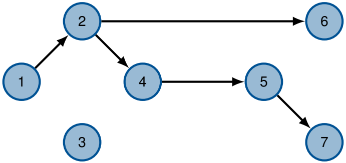

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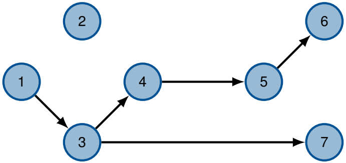

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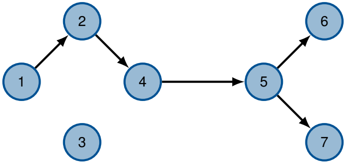

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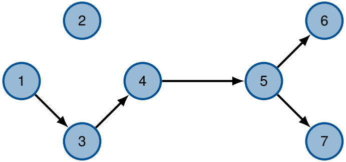

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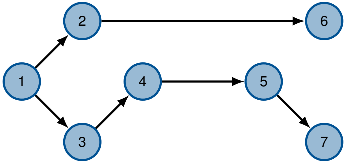

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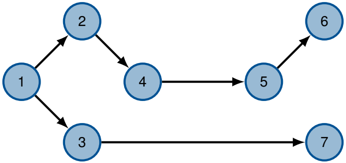

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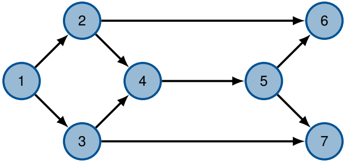

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  - (1, 2), (1, 3), (2, 4), (3, 7), (4, 5), (5, 6)
- Optimal solution is a superposition of those multicast trees
  - subject to the capacity constraints but
  - without explicit flow conservation law.

Multicast tree-based forwarding

max s–T flow problem

- Multicast trees $\mathrm{MT}_{sT} = \{1, \dots, K\}$
- Multicast tree incidence vector $\boldsymbol{x}_k$ such that $x_{k,j} = 1$ if arc j is in the k-th multicast tree, otherwise $x_{k,j} = 0$
- One commodity for the multicast, no flow conservation constraint
- Capacity vector $\boldsymbol{z}$ split among all trees

$$\begin{aligned}
\max \quad & \sum_{k \in \mathrm{MT}_{sT}} r_k \\
\text{s. t.} \quad & \sum_{k \in \mathrm{MT}_{sT}} r_k \boldsymbol{x}_k \leq \boldsymbol{z} \\
& r_k \geq 0 \quad \forall k \in \mathrm{MT}_{sT}
\end{aligned}$$

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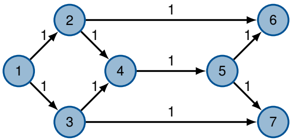
& r_k \ge 0 \quad \forall k \in \mathrm{MT}_{sT}
\end{aligned}
$$

Notes:

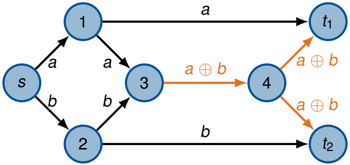
- Finding all multicast trees is a hard problem.
- In practice, heuristics that approximate optimal solutions are being used.

Multicast tree-based forwarding



- Trees in optimal solution
  - (1, 2), (1, 3), (2, 6), (3, 7) $\boldsymbol{x}_1 = [\,1\;1\;0\;1\;0\;1\;0\;0\;0\,]^{\mathrm{T}}$
  - (1, 2), (2, 4), (2, 6), (4, 5), (5, 7) $\boldsymbol{x}_2 = [\,1\;0\;1\;1\;0\;0\;1\;0\;1\,]^{\mathrm{T}}$
  - (1, 3), (3, 4), (3, 7), (4, 5), (5, 6) $\boldsymbol{x}_3 = [\,0\;1\;0\;0\;1\;1\;1\;1\;0\,]^{\mathrm{T}}$
- Each tree carries rate 0.5.
- Total flow which is capacity relevant
  - $0.5(\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3) = [\,1\;1\;0.5\;1\;0.5\;1\;1\;0.5\;0.5\,]^{\mathrm{T}}$

Maximum Multicast s–T Flow = 1.5



- A single packet (coded unit of information) may serve multiple terminals simultaneously.
- Consider flow to each terminal separately.
- But capacity is shared among all flows, i. e., each flow can use the full capacity on each arc.
- Example $(4, 5)$: flow s–t_1 and s–t_2 transmit unit of information over this arc, but only one coded packet is transmitted.

Multicast with network coding

Network coding: Max s–T flow problem

- One commodity flow $\boldsymbol{x}_t$ for each terminal $t \in T$
- Source vector $\boldsymbol{d}_{st}$ for each terminal $t \in T$
- Capacity vector $\boldsymbol{z}$ is shared for all flows, i. e., capacity on each arc can be fully exploited by each commodity flow.

$$\begin{aligned} \max\ r \quad \text{s. t.} \quad \boldsymbol{M}\boldsymbol{x}_t &= r\boldsymbol{d}_{st} && \forall t \in T \\ \boldsymbol{x}_t &\geq \boldsymbol{0} && \forall t \in T \\ \boldsymbol{x}_t &\leq \boldsymbol{z} && \forall t \in T \end{aligned}$$

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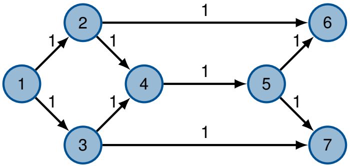
Note the difference:

- Capacity constraint must be fulfilled for individual flows only.
- There is no joint capacity constraint any more!

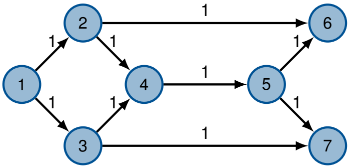


- Optimal flow solutions
  - $\boldsymbol{x}_6 = [1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1]^T$
  - $\boldsymbol{x}_7 = [1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0]^T$
- Total flow which is capacity relevant
  - $\max(\boldsymbol{x}_6, \boldsymbol{x}_7) = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$

Maximum Multicast s–T Flow = 2

Multicast with network coding

Comparison for Butterfly



mode	achievable capacity
store-forward	1.0
multicast tree	1.5
network coding	2.0

- Can we do even better?

Multicast with network coding

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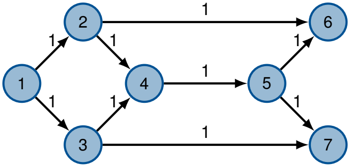

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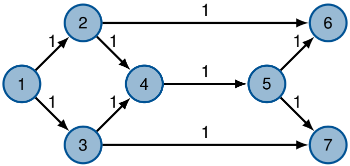

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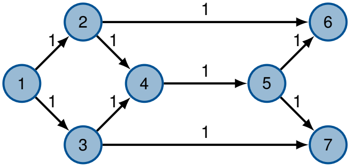

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- Can we do even better? No! Why?

Min-cut upper bound on multicast rate

- Find all s–T cuts S and their values $v(S)$.
- The cut with $v(S)$ minimal limits the maximum flow.

Max-flow min-cut theorem (reformulated)

The value of the minimum s–t cut for all terminals $t \in T$ equals the value of the maximum s–T flow with network coding, i. e.,

$$\max\{r : \boldsymbol{M}\boldsymbol{x}_t = r\boldsymbol{d}_{st},\ \boldsymbol{0} \leq \boldsymbol{x}_t \leq \boldsymbol{z},\ \forall t \in T\}$$

$$=$$

$$\min_{t \in T} \min\{v(S) : S \text{ is } s\text{–}t \text{ cut}\}$$

Wired vs. Wireless — Typical Properties

Wired Networks

- Most wired networks are composed from individual point-to-point links, which do not interact and share no resources on the physical layer.
- Physical links are almost lossless and error-free.
- Wired networks can be modeled as abstract graphs with perfect capacitated links for throughput calculation.

Wireless Packet Networks

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- Wired networks can be modeled as abstract graphs with perfect capacitated links for throughput calculation.

Wireless Networks

- Wireless networks share a common transmission medium.
- The medium is shared and omnidirectional, which turns it into a broadcast medium and causes interference.
- Wireless transmissions are prone to errors leading to packet errors or packet loss.
- How can we model wireless networks? Graphs?

Note: There are wired networks that use broadcast media, such as good old Ethernet without switches. Are there other such networks in use today?

Wireless Packet Networks

Packet Networks

- Information is encoded into packets, which are protected by an
  - error correcting code on the physical layer (channel code) for removing inevitable transmission errors and an
  - error detecting code (e. g. CRC) to detect any residual errors or decoding failures of the channel code, and
- have individual addressing information attached in order to route packets indepedently from source to destination.

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Note: on point-to-point links there may be no need for addressing information, e. g. Serial Line Internet Protocol (SLIP).

Wireless Packet Networks

- Due the broadcast nature of wireless transmissions, elaborated schemes for medium access are needed:
  - Simultaneous transmissions may cause interference.
  - Without simultaneous transmissions resources may be wasted.
  - Some kind of fairness should be provided.

  ⇒ Medium access needs to be organized (centrally or distributed).

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  - Loss may be also due to interference (packet collisions).

- Transmitted packets are not only received by one (intended) node but by multiple nodes (known as wireless broadcast advantage).
  - Need to model selective overhearing of individual packets. Who gets which packet?

Model 1: simple graph model with orthogonal medium access

- Ignore broadcast advantage, i. e., transmissions are ignored by all but the intended receiver.
- Modify arc capacities to consider
  - medium access and interference, and
  - packet losses.

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Wireless network model

- Graph (N, A)
- Arc capacity vector $\boldsymbol{z}$
- Region of admissible capacity vectors $\mathcal{Z}$:
  - Each $\boldsymbol{z} \in \mathcal{Z}$ corresponds to a different trade-off between all arcs.
  - Trade-off is necessary due to shared resources and interference.

Note: Compare to wired networks, where each arc capacity depends only on the properties of the underlying link.

Model 1: simple graph model with orthogonal medium access

Assumptions

- Same code rate for all packets
- Equal and arbitrarily fine splitting of resources
- No simultaneous transmissions (orthogonal medium access)
- No interference
- Shared transmission time / frequency resources:
  resource share τ_j of arc j s. t. total resource shares add up to 1
- Packet loss (due to fading / noise / mobility / . . .):
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Arc Capacity Region (NC or ACK/NACK)

$$\mathcal{Z} = \bigcup_{\substack{\boldsymbol{\tau} \geq \mathbf{0}: \\ \mathbf{1}^{\mathsf{T}} \boldsymbol{\tau} \leq 1}} \{\boldsymbol{z} : z_j = \tau_j(1 - \varepsilon_j)\}$$

Model 1: simple graph model with orthogonal medium access

Maximum s-t Flow

- Source vector $\boldsymbol{d}_{st}$
- Incidence matrix $\boldsymbol{M}$
- Arc capacity region $\mathcal{Z}$

$$\max \; r \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}_{st}$$
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Maximum s-T multicast flow

- Source vector $\boldsymbol{d}_{st}$ for all $t \in T$
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Hypergraphs

Directed hypergraphs

A directed hypergraph $G = (N, H)$ consists of a

- set of nodes $N = \{1, \dots, n\}$ and
- set of hyperarcs $H = \{1, \dots, m\}$ where
  - each hyperarc $j \in H$ represents an ordered pair (a, B) of
  - of a source node $a \in N$ and
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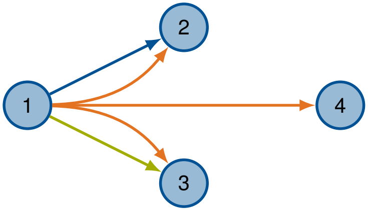
Example:

All hyperarcs $(a, B) \in \mathcal{H}$ with $a = 1$:

- $(1, \{2\})$, $(1, \{3\})$, $(1, \{4\})$
- $(1, \{2, 3\})$, $(1, \{2, 4\})$, $(1, \{3, 4\})$
- $(1, \{2, 3, 4\})$



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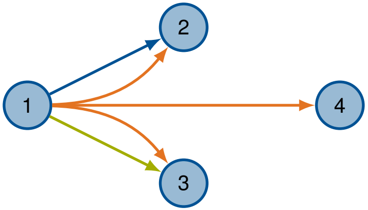

If a packet is sent over hyperarc $j \equiv (a, B)$, then

- all nodes $b \in B$ overhear an (identical) copy of that packet and
- no other node $i \notin B$ overhears that packet.

Hypergraphs

The directed graph (N, A) induced by hypergraph (N, H) consists of

- all arcs $k \equiv (a, b)$ such that
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For the induced graph we

- denote the set of arcs that is generated by hyperarc j by A_j and
- define the hyperarc-arc incidence matrix as

$$\boldsymbol{N} = (n)_{jk} = \begin{cases} 1 & \text{if } k \in A_j, \\ 0 & \text{otherwise.} \end{cases}$$

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Example:

$j \in H$	$(a, B) \in \mathcal{H}$	A_j	$k \in A$
1	$(1, \{2\})$	$\{(1, 2)\}$	1
2	$(1, \{3\})$	$\{(1, 3)\}$	2
3	$(1, \{4\})$	$\{(1, 4)\}$	3
4	$(1, \{2, 3\})$	$\{(1, 2), (1, 3)\}$	1,2
5	$(1, \{2, 4\})$	$\{(1, 2), (1, 4)\}$	1,3
6	$(1, \{3, 4\})$	$\{(1, 3), (1, 4)\}$	2,3
7	$(1, \{2, 3, 4\})$	$\{(1, 2), (1, 3), (1, 4)\}$	1,2,3

Model 2: lossless hypergraph model with orthogonal medium access

- Hypergraph (N, H)
- One hyperarc per node (simplification), enumerated according to the node the hyperarc is originating at
- Inherits MAC properties from model 1 (orthogonal medium access)
- Each node gets a resource share $\tau_i \geq 0$ such that $\sum_{i \in N} \tau_i \leq 1$
- Packets transmitted on a hyperarc $j \equiv (a, B)$ are received by all nodes $b \in B$
- No packets are lost

Model 2: lossless hypergraph model with orthogonal medium access

Information flow in lossless hypergraphs (model 2)

- Information flow vector $\boldsymbol{x}$ on induced graph (N, A)
- Demand vector $\boldsymbol{d}$ and incidence matrix $\boldsymbol{M}$
- Flow must be conserved on induced graph, i. e.,

$$\boldsymbol{M}\boldsymbol{x} = \boldsymbol{d}$$

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- Hyperarc-arc incidence matrix $\boldsymbol{N}$
- Receivers of a hyperarc get identical packets over this hyperarc
- Each piece of information can only be used once (by one node)

$$\boldsymbol{Nx} \leq \boldsymbol{z} \quad \Leftrightarrow \quad \sum_{k \in A_j} x_k \leq z_j \quad \forall j \in H$$

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- Lossless Hyperarc Capacity Region (NC)

$$\mathcal{Z} = \bigcup_{\substack{\boldsymbol{\tau} \geq \boldsymbol{0}: \\ \boldsymbol{1}^\mathrm{T} \boldsymbol{\tau} \leq 1}} \{\boldsymbol{z} : z_j = \tau_j\}$$

Model 2: lossless hypergraph model with orthogonal medium access

Hyperarc maximum s-t flow (routing/network coding)

- Source vector $\boldsymbol{d}_{st}$
- Incidence matrix $\boldsymbol{M}$
- Hyperarc-arc incidence matrix $\boldsymbol{N}$
- Hyperarc capacity region $\mathscr{Z}$

$$\begin{aligned} \max\ r \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x} &= r\boldsymbol{d}_{st} \\ \boldsymbol{x} &\geq \boldsymbol{0} \\ \boldsymbol{N}\boldsymbol{x} &\leq \boldsymbol{z} \\ \boldsymbol{z} &\in \mathscr{Z} \end{aligned}$$

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Hyperarc maximum *s*-*t* flow (routing/network coding)

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Hyperarc maximum *s*-*T* multicast flow (network coding)

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- Hyperarc-arc incidence matrix $\boldsymbol{N}$
- Hyperarc capacity region $\mathcal{Z}$

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Note

- We can use each hyperarc (packet) only once for each terminal.
- But we can use each hyperarc differently for each terminal.

Model 2: lossless hypergraph model with orthogonal medium access

Hyperarc min-cut model

- An s-t cut is a subset of nodes $S \subset N$ such that $s \in S$ and $t \notin S$.
- A hyperarc $j \equiv (a, B) \in \mathcal{H}$ crosses S if $a \in S$ and $B \not\subset S$.
  $\mathcal{H}(S)$ denotes all crossing arcs, and $H(S)$ their indices.
- The value of any s-t cut upper bounds the maximum s-t flow.
- The value of an s-t cut given the capacity vector $\boldsymbol{z}$ is defined as

$$v(S) = \sum_{j \in H(S)} z_j = \sum_{(a,B) \in \mathcal{H}(S)} z_{aB}$$

- Model 2 (only one hyperarc per node, $z_{aB} = \tau_a$):

$$v(S) = \sum_{(a,B) \in \mathcal{H}(S)} \tau_a .$$

Model 2: lossless hypergraph model with orthogonal medium access

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Model 3: lossy hypergraph model with orthogonal medium access

Idea: Use model 2 but

- account for packet erasures and
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$\Rightarrow$ one hyperarc per node is insufficient

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Changes made to model 2:

- Hypergraph (N, H) with induced graph (N, A).
- Let N_a denote the set of neighbors of node $a \in N$.
- For each node $a \in N$ consider all possible hyperarcs $j \equiv (a, B)$ for any $B \subset N_a$.
- Packet loss is independent across all receivers (simplification).
- Packets from a to b are lost with probability ε_k where $k \equiv (a, b)$, i. e., $k \in A$ is an arc index of the induced graph.
- A packet transmitted by $a \in N$ is transmitted on hyperarc $j \equiv (a, B)$, i. e., it is received precisely by $B \subset N_a$ and lost by all other nodes $N_a \setminus B$, with probability

$$\Pr[\text{“no loss on } j \equiv (a, B)\text{”} \mid \text{“}a \text{ transmits”}] = \prod_{b \in B} (1 - \varepsilon_{ab}) \prod_{b \notin B} \varepsilon_{ab}.$$

Model 3: lossy hypergraph model with orthogonal medium access

Information flow in lossy hypergraphs (model 3)

- Information flow vector $\boldsymbol{x}$ on induced graph (N, A).
- Flow must be conserved on induced graph, i. e., $\boldsymbol{Mx} = \boldsymbol{d}$.
- Receivers of a hyperarc get identical packets over this hyperarc provided they have not lost the packets.

Given a transmitter a, we are interested in an upper bound for the flow from a to a set of receivers $B \subset N_a$ (where N_a denotes the neighborhood of node a).

- Each piece of information can only be used by one successful receiver.
- The total flow from a to B must not exceed the total amount of different received packets of this set of nodes.

This is equivalent to the probability that

- node a is transmitting at all and
- at least one node $b \in B$ overhears the transmission.

It does not matter whether or not more than one or which specific node in B overhears the transmission.

Model 3: lossy hypergraph model with orthogonal medium access

For a 1-receiver set $B = \{b\}$ and induced arc $k \equiv (a, b)$:

- Which hyperarcs may transport packets from a to b?

$\Rightarrow$ Flow bound:

$$x_k = x_{ab} \leq$$

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For a 1-receiver set $B = \{b\}$ and induced arc $k \equiv (a, b)$:

- Which hyperarcs may transport packets from a to b?

  Any hyperarc $j' \equiv (a, B')$ with $b \in B'$, and precisely these hyperarcs induce arc $k \equiv (a, b)$.

  $\Rightarrow$ Consider all $(a, B') \equiv j' \in H : (a, b) \equiv k \in A_{j'}$.

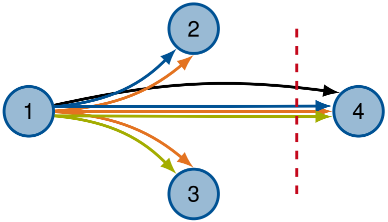

Figure 1: Example for $a = 1$ and $b = 4$, only hyperarcs $j' \in H : k \in A_{j'}$ are shown

$\Rightarrow$ Flow bound:

$$x_k = x_{ab} \leq \sum_{\substack{(a,B')\equiv j':\\ (a,b)\in A_{j'}}}$$

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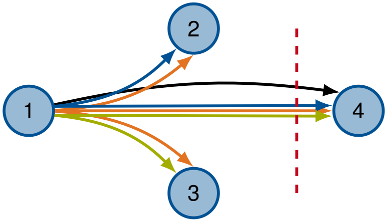

Figure 1: Example for $a = 1$ and $b = 4$, only hyperarcs $j' \in H : k \in A_{j'}$ are shown

- Packets may be transferred over any of these hyperarcs $(a, B') \equiv j'$, but only if node a is transmitting at all.

$\Rightarrow$ Flow bound:

$$x_k = x_{ab} \leq \sum_{\substack{(a,B')\equiv j' : \\ (a,b)\in A_{j'}}} \tau_a$$

Model 3: lossy hypergraph model with orthogonal medium access

For a 1-receiver set $B = \{b\}$ and induced arc $k \equiv (a, b)$:

- Which hyperarcs may transport packets from a to b?

  Any hyperarc $j' \equiv (a, B')$ with $b \in B'$, and precisely these hyperarcs induce arc $k \equiv (a, b)$.

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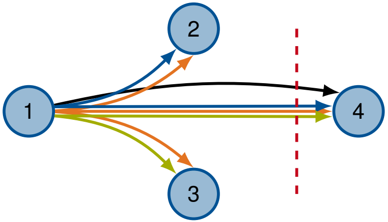

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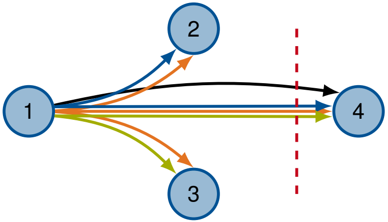

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Model 3: lossy hypergraph model with orthogonal medium access

For a 2-receiver set $B = \{b_1, b_2\}$ and induced arcs $k_1 \equiv (a, b_1)$ and $k_2 \equiv (a, b_2)$:

- Which $j' \in H$ may transport packets to either b_1 or b_2?

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$$x_{k_1} + x_{k_2} = x_{ab_1} + x_{ab_2} \leq$$

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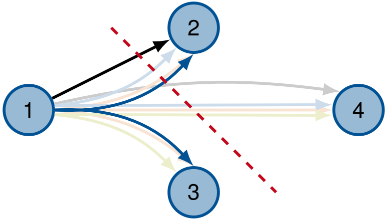

Figure 2: Example for $a = 1$ and $B = \{2, 4\}$; shaded hyperarcs are for $B' = \{4\}$, solid hyperarcs are the additions for $B = B' \cup \{2\}$

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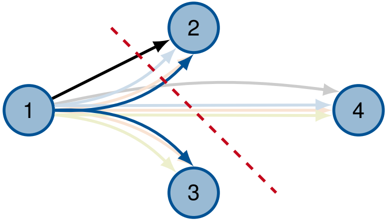

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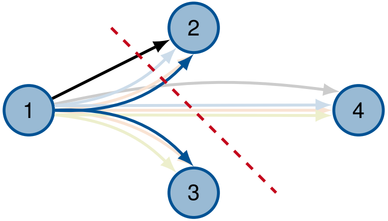

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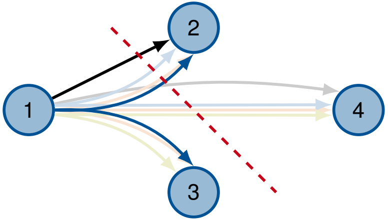

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Model 3: lossy hypergraph model with orthogonal medium access

Generalization to multiple receiver sets:

- Each pair (a, B) corresponds to some hyperarc j, i. e., $j \equiv (a, B)$.
- That hyperarc induces the set A_j of arcs.
- The flow bound is determined by all hyperarcs $j' \in H : A_j \cap A_{j'} \neq \emptyset$:

$$\sum_{k \in A_j} x_k = \sum_{(a,b) \in \mathcal{A}_j} x_{ab} \leq \sum_{\substack{(a,B') \equiv j': \\ A_j \cap A_{j'} \neq \emptyset}} \tau_a \prod_{b' \in B'} (1 - \varepsilon_{ab'}) \prod_{b' \notin B'} \varepsilon_{ab'} = y_{aB} = y_j$$

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$$z_j = z_{aB} = \tau_a \prod_{b \in B} (1 - \varepsilon_{ab}) \prod_{b \notin B} \varepsilon_{ab}$$

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- Hyperarc capacity region:

$$\mathcal{Z} = \bigcup_{\substack{\boldsymbol{\tau} \geq \mathbf{0} \\ \mathbf{1}^{\mathrm{T}} \boldsymbol{\tau} \leq 1}} \left\{ \mathbf{z} : z_j = z_{aB} = \tau_a \prod_{b \in B} (1 - \varepsilon_{ab}) \prod_{b \notin B} \varepsilon_{ab} \quad \forall j \equiv (a, B) \in \mathcal{H} \right\}$$

Model 3: lossy hypergraph model with orthogonal medium access

Information flow in lossy hypergraphs (model 3)

- Reformulation of the lossy flow bound (for all receiver set):

$$\sum_{k \in A_j} x_k \leq \tau_a \left(1 - \prod_{k \in A_j} \varepsilon_k \right) = y_j \quad \forall j \equiv (a, B) \in \mathcal{H}$$

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- Broadcast capacity region

$$\mathcal{Y} = \bigcup_{\substack{\boldsymbol{\tau} \geq \mathbf{0} \\ \mathbf{1}^\mathrm{T} \boldsymbol{\tau} \leq 1}} \left\{ \boldsymbol{y} : y_j = \tau_a \left(1 - \prod_{k \in A_j} \varepsilon_k\right) \quad \forall j \equiv (a, B) \in \mathcal{H} \right\}$$

Model 3: lossy hypergraph model with orthogonal medium access

Information flow in lossy hypergraphs (model 3)

- Hyperarc-arc incidence matrix

$$\boldsymbol{N} = (N_{jk}) = \begin{cases} 1 & \text{if } k \in A_j \\ 0 & \text{otherwise} \end{cases}$$

- Hyperarc-hyperarc incidence matrix

$$\boldsymbol{Q} = (Q_{ij}) = \begin{cases} 1 & \text{if } A_i \cap A_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- Hyperarc-to-broadcast transformation

$$\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{z}$$

- Lossy hyperarc flow bound with hyperarc capacity region

$$\boldsymbol{N}\boldsymbol{x} \leq \boldsymbol{Q}\boldsymbol{z}$$

- Lossy hyperarc flow bound with broadcast capacity region

$$\boldsymbol{N}\boldsymbol{x} \leq \boldsymbol{y}$$

Model 3: lossy hypergraph model with orthogonal medium access

Lossy hyperarc maximum s-t flow (opportunistic RT/NC)

- Source vector $\boldsymbol{d}_{st}$
- Incidence matrix $\boldsymbol{M}$
- Hyperarc-arc incidence matrix $\boldsymbol{N}$
- Broadcast capacity region $\mathcal{Y}$

$$
\begin{aligned}
\max \quad r \quad \text{s.t.} \quad & \boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}_{st} \\
& \boldsymbol{x} \geq \boldsymbol{0} \\
& \boldsymbol{N}\boldsymbol{x} \leq \boldsymbol{y} \\
& \boldsymbol{y} \in \mathcal{Y}
\end{aligned}
$$

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Lossy hyperarc maximum s-T multicast flow (NC)

- Source vector $\boldsymbol{d}_{st}$
- Incidence matrix $\boldsymbol{M}$
- Hyperarc-arc incidence matrix $\boldsymbol{N}$
- Broadcast capacity region $\mathcal{Y}$

$$\begin{aligned} \max \; r \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x}_t &= r\boldsymbol{d}_{st} && \forall t \in T \\ \boldsymbol{x}_t &\geq \boldsymbol{0} && \forall t \in T \\ \boldsymbol{N}\boldsymbol{x}_t &\leq \boldsymbol{y} && \forall t \in T \\ \boldsymbol{y} &\in \mathcal{Y} \end{aligned}$$

Model 3: lossy hypergraph model with orthogonal medium access

Lossy hyperarc min-cut model

- An s-t cut is a subset of nodes $S \subset N$ such that $s \in S$ and $t \notin S$.
- A hyperarc $j \equiv (a, B) \in \mathcal{H}$ crosses S if $a \in S$ and $B \not\subset S$, i. e., $B \cap (N \setminus S) \neq \emptyset$.
- $H(S)$ denotes all crossing hyperarc indices, $\mathcal{H}(S)$ all crossing hyperarcs.
- The value of any s-t cut upper bounds the maximum s-t flow.
- The value of an s-t cut given the capacity vector $\boldsymbol{z}$ is defined as

$$v(S) = \sum_{j \in H(S)} z_j.$$

- Cut value of model 3:

$$v(S) = \sum_{(a,B) \in \mathcal{H}(S)} \tau_a \prod_{b \in B} (1 - \varepsilon_{ab}) \prod_{b \notin B} \varepsilon_{ab}.$$

- $\mathcal{A}_a(S)$: Set of arcs $(a, b) \in \mathcal{A} : b \in N \setminus S$
  ($A_a(S)$ denotes index set of crossing arcs)
- $\mathcal{H}_a(S)$: Set of hyperarcs $(a, B) \in \mathcal{H} : B \cap (N \setminus S) \neq \emptyset$
  ($H_a(S)$ denotes the index set of crossing hyperarcs)

Model 3: lossy hypergraph model with orthogonal medium access

Lossy hyperarc min-cut model

- Characterize $\mathcal{H}(S)$:

$$\begin{aligned}\mathcal{H}(S) &= \{(a,B) \in \mathcal{H} : a \in S, B \cap (N \setminus S) \neq \emptyset\} \\ &= \bigcup_{a \in S} \mathcal{H}_a(S) \\ &= \bigcup_{a \in S} \{(a,B) \equiv j \in H : A_j \cap A_a(S) \neq \emptyset\}\end{aligned}$$

- Cut value of Model 3 (looks very much like flow bound):

$$v(S) = \sum_{a \in S} \sum_{\substack{j \equiv (a,B): \\ A_j \cap A_a(S) \neq \emptyset}} \tau_a \prod_{b \in B} (1 - \varepsilon_{ab}) \prod_{b \notin B} \varepsilon_{ab}$$

Model 3: lossy hypergraph model with orthogonal medium access

Lossy hyperarc min-cut model

- Flow bound:

$$\sum_{k \in A_j} x_k \leq \sum_{\substack{j' \equiv (a,B'): \\ A_j \cap A_{j'} \neq \emptyset}} \tau_a \prod_{b' \in B'} (1 - \varepsilon_{ab'}) \prod_{b' \notin B'} \varepsilon_{ab'} = y_j \quad \forall j \equiv (a, B) \in \mathcal{H}$$

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Multicast max-flow min-cut theorem (model 3)

The value of the minimum s-t cut for all terminals $t \in T$ equals the value of the maximum s-T flow with network coding, i. e.,

$$\max\left\{r : \boldsymbol{M}\boldsymbol{x}_t = r\boldsymbol{d}_{st},\ \boldsymbol{0} \leq \boldsymbol{x}_t,\ \boldsymbol{N}\boldsymbol{x}_t \leq \boldsymbol{y},\ \forall t \in T, \boldsymbol{y} \in \mathcal{Y}\right\}$$

$$=$$

$$\max_{\boldsymbol{y}\in\mathcal{Y}} \min_{t\in T} \min\left\{v(S) = \sum_{\substack{j\equiv(a,B)\in\mathcal{H}: \\ a\in S\wedge B=N_a\setminus S}} y_j : S \text{ is } s\text{-}t \text{ cut}\right\}$$

Model overview

Section 1: Model 1 – lossy non-hypergraph model

- Reflects lossy wired networks
- Cannot intuitively cope with broadcast media
- Flow bound: $\boldsymbol{x} \leq \boldsymbol{z} \quad \mathcal{Z} = \bigcup_{\boldsymbol{\tau} \geq \mathbf{0} \wedge \mathbf{1}^\mathsf{T} \boldsymbol{\tau} \leq 1} \{\boldsymbol{z} : z_k = \tau_k(1 - \varepsilon_k) \quad \forall k \in A\}$

Note: Model 1 $\subset$ Model 2 $\subset$ Model 3

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Section 3: Model 2 – lossless hypergraph

- Considers broadcast media by hyperarcs
- No losses, i. e., single hyperarc from some $a \in N$ to all its neighbors $b \in N_a$
- Flow bound: $\boldsymbol{N}\boldsymbol{x} \le \boldsymbol{z} \quad \mathcal{Z} = \bigcup_{\boldsymbol{\tau} \ge \boldsymbol{0} \wedge \boldsymbol{1}^\mathsf{T}\boldsymbol{\tau} \le 1} \{\boldsymbol{z} : z_j = \tau_j \quad \forall j \in H\}$

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Section 4: Model 3 – lossy hypergraph

- Considers broadcast media by hyperarcs
- Allows for losses, i. e., hyperarcs from $a \in N$ to all subsets $B \subset N_a$ of neighbors
- Flow bound: $\boldsymbol{N x} \leq \boldsymbol{Q z} = \boldsymbol{y} \quad \mathcal{Y} = \bigcup_{\boldsymbol{\tau} \geq \boldsymbol{0} \wedge \boldsymbol{1}^{\mathrm{T}} \boldsymbol{\tau} \leq 1} \left\{\boldsymbol{y} : y_j = \tau_a \left(1 - \prod_{k \in A_j} \varepsilon_k\right) \quad \forall j \equiv (a, B) \in \mathcal{H}\right\}$

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